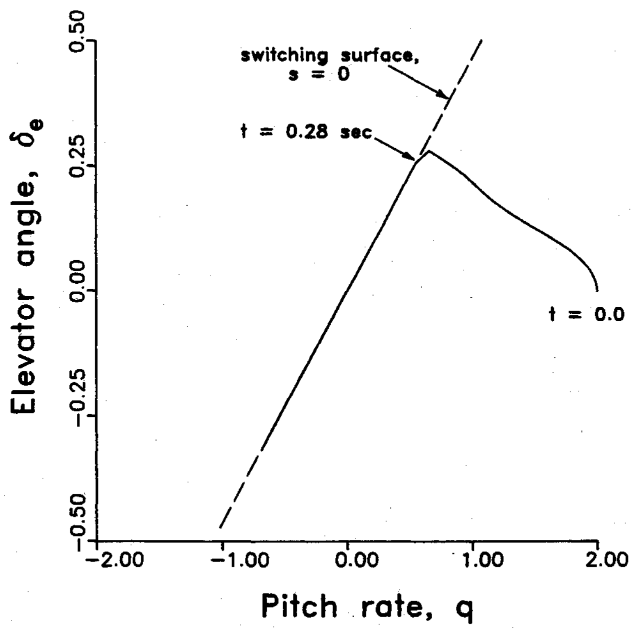

Fig. 3 Elevator angle vs pitch rate, trajectory in measurement space.

$s \simeq 0$ at this time. It should be noted that once sliding has occurred, the system behaves with second-order dynamics having eigenvalues specified by Eq. (22). In the sliding mode, the third state δ_e is simply given by a linear combination of α and q. Since $s = GC_1x_1 + GC_2x_2 = 0$,

$$\delta_e = -(GC_2)^{-1}GC_1\begin{bmatrix}\alpha\\ q\end{bmatrix} = 0.4635q \tag{24}$$

In Fig. 3, the system trajectory is plotted in the measurement space y_2 vs y_1. The trajectory is seen to converge to the line $Gy = 0$, shown dashed in the figure.

V. Conclusions

In this Note, output feedback has been successfully applied to variable structure systems. The method allows one to use sliding-mode control theory on systems for which full-state feedback or estimated state feedback is not possible or not desirable. Output feedback allows for much simpler implementation, while at the same time providing good robustness to disturbance and plant uncertainty. Like output feedback in linear systems, it is seen that output feedback in variable structure systems is somewhat more restrictive and is harder to design than estimated state feedback with an asymptotic observer. It is also more sensitive to noise due to the direct link to the measurements. The ease of implementation, though, makes output feedback a viable alternative to either state feedback or estimated state feedback for many practical applications of variable structure systems.

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Dynamics of a Rotationally Accelerated Beam

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Introduction

FLEXIBLE elements in non-steady-state rotation frequently constitute an important part of a larger mechanism. For example, a key area of research in robotics is the design of lightweight manipulators having quick response with modest energy input and whose dynamic response is not significantly affected by structural flexibility.

The work presented herein addresses the problem of finding the total displacement (that due to rigid-body motion plus flexibility effects) of a cantilevered beam subject to an arbitrary driving torque. The problem is formulated in a nondimensional manner and, thus, the solution is general. The results for a harmonic-applied torque are given explicitly in order to provide the basis for a Fourier synthesis of a more general torque expression.

Algorithms for analyzing the behavior of a cantilever beam built into a rigid base that performs specified motions in three dimensions have been developed by Kane et al.[1] Their paper also contains a bibliography of the extensive literature in this research area. In the context of robot dynamics the control of the beam dynamics has been investigated by, for example, Cannon and Schmitz,[2] Chassiakos and Bekey,[3] Hastings and Book,[4] and Rakhsha and Goldenberg.[5] In addition, Sakawa et al.[6] investigate the problem of a flexible arm rotated by a motor about an axis through the arm's fixed end. Several satisfactory experimental results are given. In the context of flexible space structures the control problem is addressed in a forthcoming paper by Wie and Bryson.[7] Their work was made known to the authors after the present paper had been completed and submitted for publication.

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Physical Problem and Analytical Model

The physical problem consists of the cantilevered beam shown in Fig. 1 subject to a time-dependent torque. It is a continuous, uniform, clamped-free Bernoulli beam of length L, cross-sectional area S, density ρ, and Young's modulus E. The applied torque is $T(t)$; $\theta(t)$ is the angle through which the encastre end of the beam is rotated. The dynamic effect of the hub is represented by the lumped moment of inertia I_H.

The equations of motion can be established directly from Hamilton's principle, neglecting longitudinal extension of the beam, in the form

$$\delta\int_{t_1}^{t_2}\left[\int_0^L \frac{1}{2}\rho S\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{\partial v}{\partial t}\right)^2\mathrm{d}x+\frac{1}{2}I_H\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 -\int_0^L\frac{1}{2}EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2\mathrm{d}x-\theta T(t)\right]\mathrm{d}t=0 \tag{1}$$

where I is the area moment of inertia of the beam cross section. This statement, which neglects terms in $v^2\,(\mathrm{d}\theta/\mathrm{d}t)^2$, leads to the two equations

$$\rho S\frac{\partial^2 v}{\partial t^2}+EI\frac{\partial^4 v}{\partial x^4}+\rho Sx\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=0 \tag{2}$$

and

$$(I_B+I_H)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\int_0^L\rho Sx\frac{\partial^2 v}{\partial t^2}\,\mathrm{d}x=T(t) \tag{3}$$

where I_B is the mass moment of inertia of the beam about an end. On multiplying Eq. (2) by x, and integrating from $x=0$ to $x=L$, the integral in Eq. (3) can be eliminated to find Eq. (6). This derivation assumes, as noted previously, that the hub radius is negligible or, equivalently, that the shear force at the clamped end of the beam exerts no torque about the hub center. This restriction, made here for simplicity, can be relaxed at a slight expense in the complexity of the analysis.

Thus, the initial-value problem for the solution of $v(x,t)$ is: Solve Eq. (2) for $x\in(0,L)$ subject to the boundary conditions for $t>0$:

$$v(0,t)=\frac{\partial v}{\partial x}(0,t)=\frac{\partial^2 v}{\partial x^2}(L,t)=\frac{\partial^3 v}{\partial x^3}(L,t)=0 \tag{4}$$

and initial conditions for $x\in[0,L]$

$$v(x,0)=\phi(x),\qquad \frac{\partial v}{\partial t}(x,0)=\psi(x) \tag{5}$$

where

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=\frac{1}{I_H}\left[T(t)+EI\frac{\partial^2 v}{\partial x^2}(0,t)\right] \tag{6}$$

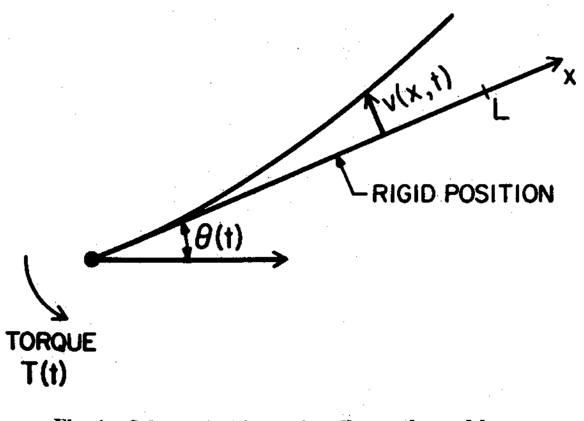

Fig. 1 Schematic of rotationally accelerated beam.

Set $\xi=x/L$, $\eta=v/L$, $\tau=\alpha t$, and $T(t)=T_0\,\vartheta(t)$, where ξ,η,τ, and $\vartheta(t)$ are the dimensionless coordinate, deflection, time, and torque, respectively, and

$$\alpha=[(EI)/(\rho SL^4)]^{1/2},\qquad \gamma=T_0/(I_H\alpha^2)$$

$$\epsilon=3I_B/I_H,\qquad I_B=(1/3)\rho SL^3$$

Note that γ and ϵ are dimensionless quantities. Then Eqs. (2) and (4-6) in dimensionless form are

$$\eta''''+\ddot\eta+\xi\ddot\theta=0,\qquad \xi\in(0,1) \tag{7}$$

subject to the boundary conditions for $\tau>0$,

$$\eta(0,\tau)=\eta'(1,\tau)=\eta''(1,\tau)=\eta'''(1,\tau)=0 \tag{8}$$

and initial conditions for $\xi\in[0,1]$

$$\eta(\xi,0)=\tilde\phi(\xi),\qquad \dot\eta(\xi,0)=\tilde\psi(\xi) \tag{9}$$

where

$$\ddot\theta(\tau)=\gamma\vartheta(\tau)+\epsilon\eta''(0,\tau) \tag{10}$$

The primes denote differentiation with respect to ξ, the dots differentiation with respect to τ, $\tilde\phi(\xi)=\phi(\xi)/L$, and $\tilde\psi(\xi)=\psi(\xi)/(\alpha L)$.

Forced Response and General Solution

To determine the forced response $V_F(\xi,\omega)$ of the system, an input torque $\vartheta(\tau)=e^{i\omega\tau}$ is considered, Equations (7) and (10) then lead, assuming an identical time dependence in $\theta(\tau)$ and $\eta(\xi,\tau)$, to

$$V''''-\omega^2V=-\xi\gamma-\epsilon\xi V''(0) \tag{11}$$

where $V(\xi)$ represents the spatial part of the displacement $\eta(\xi,\tau)$. The solution of Eq. (11), being the sum of a particular integral and the homogeneous solution, is

$$V_F(\xi,\omega)=A\cosh\lambda\xi+B\sinh\lambda\xi+C\cos\lambda\xi+D\sin\lambda\xi+\xi[\gamma+\epsilon\lambda^2(A-C)]/\omega^2 \tag{12}$$

in which $\lambda^4=\omega^2$. The application of the boundary conditions (8) determines the constants A, B, C, and D and, hence, the response to the periodic input. Their values are

$$A=r\,(\sin\lambda\cosh\lambda-\cos\lambda\sinh\lambda)=-C \tag{13a}$$

$$B=r\,(1+\cos\lambda\cosh\lambda-\sin\lambda\sinh\lambda) \tag{13b}$$

$$D=r\,(1+\cos\lambda\cosh\lambda+\sin\lambda\sinh\lambda) \tag{13c}$$

the notation $r=-\gamma/[\omega^2\Delta(\lambda)]$ and

$$\Delta(\lambda)=\frac{2\epsilon}{\omega}(\sin\lambda\cosh\lambda-\cos\lambda\sinh\lambda)+2\lambda(1+\cos\lambda\cosh\lambda) \tag{14}$$

being used. The forced response to an arbitrary input torque $\vartheta(\tau)$ can now be found using standard methods. It is seen from Eq. (12) that the response consists of a functional form commonly associated with beam vibrations on which a rigid-body displacement (rotation) is superposed.

The homogeneous solution of Eq. (11) takes the form

$$V_H(\xi)=\sum_{n=1}^{\infty}P_nV_n(\xi,\lambda_n) \tag{15}$$

in which, the P_n being arbitrary constants,

$$V_n(\xi,\lambda_n) = A_n \cosh\lambda_n\xi + B_n \sinh\lambda_n\xi + C_n \cos\lambda_n\xi + D_n \sin\lambda_n\xi + \epsilon\xi(A_n - C_n)/\lambda_n^2 \tag{16}$$

The values of λ_n are the roots of the transcendental equation

$$\Delta(\lambda_n) = \frac{2\epsilon}{\lambda_n^2}(\sin\lambda_n \cosh\lambda_n - \cos\lambda_n \sinh\lambda_n) + 2\lambda_n(1 + \cos\lambda_n \cosh\lambda_n) = 0 \tag{17}$$

and the constants A_n, B_n, C_n, D_n can easily be written down as

$$A_n = -D_n \frac{(\sinh\lambda_n + \sin\lambda_n)\lambda_n^3}{[2\epsilon \sinh\lambda_n - (\cosh\lambda_n + \cos\lambda_n)\lambda_n^3]} = -C_n$$

$$B_n = -D_n - 2A_n\epsilon/\lambda_n^3$$

It is important to note that the eigenfunctions $V_n(\xi,\lambda_n)$ are not an orthonormal set. It is possible, in fact, to show that

$$\int_0^1 V_m(\xi,\lambda_m)\, V_n(\xi,\lambda_n)\, \mathrm{d}\xi = \frac{\epsilon V_m''(0,\lambda_m)V_n''(0,\lambda_n)}{\omega_m^2\omega_n^2}\left(\frac{\epsilon}{3} + 1\right) \tag{18}$$

In the particular case $\epsilon = 0$, the eigenfunctions $V_n(\xi,\lambda_n)$ are mutually orthogonal and coincide with those of a fixed cantilever beam, viz.,

$$W_n(\xi,\mu_n) = Q_n\left[-\frac{(\sinh\mu_n + \sin\mu_n)}{(\cosh\mu_n + \cos\mu_n)}(\cosh\mu_n\xi - \cos\mu_n\xi) + \sinh\mu_n\xi - \sin\mu_n\xi\right] \tag{19}$$

where Q_n are normalizing constants, and μ_n are determined by the frequency equation $\cos\mu_n \cosh\mu_n = -1$, which is the limit of Eq. (17) as $\epsilon \to 0$. Because of the nonorthogonality already referred to, it is not possible to match a solution to a given initial displacement and velocity distribution in the usual direct fashion. However, it is feasible to proceed indirectly as follows.

Using the cantilever eigenfunctions $W_n(\xi,\mu_n)$, one expands the forced response in the series

$$V_F(\xi,\omega) = \sum_{j=1}^{\infty} F_j W_j(\xi,\mu_j) \tag{20}$$

and each of the eigenfunctions $V_n(\xi,\lambda_n)$ in a similar form

$$V_n(\xi,\lambda_n) = \sum_{k=1}^{\infty} G_k^{(n)} W_k(\xi,\mu_k) \tag{21}$$

Consequently, the general solution of Eq. (11) corresponding to an input torque of frequency ω assumes the form

$$V(\xi,\omega) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} E_i G_k^{(n)} W_k(\xi,\mu_k) + \sum_{j=1}^{\infty} F_j W_j(\xi,\mu_j) \tag{22}$$

In Eq. (22), the constants $G_k^{(n)}$ and F_j are known, whereas the constants E_i are to be determined by the initial displacement distribution. Explicitly, one finds for a distribution

$$\tilde{\phi}(\xi) = \sum_{m=1}^{\infty} H_m W_m(\xi,\mu_m) \tag{23}$$

that

$$H_s = \sum_{i=1}^{\infty} E_i G_s^{(i)} + F_s, \qquad s = 1,2,\ldots \tag{24}$$

In deriving Eq. (24), it is assumed that the cantilever eigenfunctions $W_m(\xi,\mu_m)$ are normalized to unity. Hence, the constants $E_i (i = 1,2,\ldots)$ are determined by the infinite set of linear algebraic equations

$$\sum_{i=1}^{\infty} E_i G_s^{(i)} = H_s - F_s, \qquad s = 1,2,\ldots \tag{25}$$

Angular Position of Arm

The rigid-body acceleration of the arm is found on using Eqs. (12) and (15) in Eq. (10) to be

$$\ddot{\theta} = \gamma e^{i\omega\tau} + \epsilon\omega(A - C)e^{i\omega\tau} + \epsilon\sum_{n=1}^{\infty} \bar{E}_n\omega_n(A_n - C_n)e^{i\omega_n\tau} \tag{26}$$

where $\bar{E}_n = E_n/D_n$, from which the angular position can be determined directly.

Numerical Example

The following example is used to obtain numerical results. The properties of the beam and hub are: Both are aluminum with $E = 10.6 \times 10^6$ lbf/in.^{2}; $\rho = 2.5879 \times 10^{-4}$ lbm/in.^{3}; beam dimensions: 10 in. $\times$ 1 in. $\times$ 0.1 in.; $I = 8.33333 \times 10^{-5}$ in.^{4}; $I_H = 6.50433 \times 10^{-3}$ lbm in.^{2}; $T_0 = 1.75398$ lbf in. The initial displacement distribution was taken to be

$$\tilde{\phi}(\xi) = \frac{a}{2\epsilon}\xi^2 - \frac{a(1+2\epsilon)}{3\epsilon}\xi^3 + \frac{a(1+8\epsilon)}{12\epsilon}\xi^4 - \frac{a}{5}\xi^5$$

where $a = \ddot{\theta}(0) = 5\ g/(100\ L)$ and $\epsilon = 3.978877$. The torque input frequency was taken as $\omega = 0.1\ \lambda_1^2$. The resulting natural frequencies and coefficients are presented in Tables 1 and 2. These results were obtained using an IBM PC-AT and the False Position Iteration Method from the Fortran subroutine software available for personal computers. The program was run in double precision with convergence occurring rapidly.

Table 1 Natural frequencies and series coefficients

r	λ_r	μ_r	F_r	H_r	$\bar{E}_r$
1	2.293159	1.875104	1.578451E − 03	0.114144E + 00	− 1.468414E − 02
2	4.733087	4.694091	6.273311E − 06	− 1.982737E − 02	1.824676E − 02
3	7.863009	7.854757	2.856231E − 07	− 1.736165E − 03	1.720129E − 03
4	10.998540	10.995541	3.795486E − 08	− 3.442806E − 04	3.446449E − 04
5	14.138578	14.137168	8.402382E − 09	− 1.013055E − 04	1.015783E − 04
6	17.279531	17.278759	2.514016E − 09	− 3.794098E − 05	3.808828E − 05

Table 2 Eigenfunction series coefficients

s	$G_s^{(1)}$	$G_s^{(2)}$	$G_s^{(3)}$	$G_s^{(4)}$	$G_s^{(5)}$	$G_s^{(6)}$
1	− 7.94886E + 00	− 0.219908E + 00	− 0.747315E − 01	− 0.376754E − 01	− 0.227153E − 01	− 0.151858E − 01
2	0.423575E − 01	− 1.05145E + 00	− 0.136156E − 01	− 0.621287E − 02	− 0.366809E − 02	− 0.243609E − 02
3	0.183291E − 02	0.185627E − 02	− 1.01285E + 00	− 0.289948E − 02	− 0.143036E − 02	− 0.903933E − 03
4	0.242267E − 03	0.221774E − 03	0.767112E − 03	− 1.00394E + 00	− 0.104120E − 02	− 0.529335E − 03
5	0.535673E − 04	0.480126E − 04	0.138680E − 03	0.382817E − 03	− 1.00177E + 00	− 0.428056E − 03
6	0.160725E − 04	0.143091E − 04	0.392491E − 04	0.864317E − 03	0.154205E − 03	− 0.999823E + 00

Summary and Conclusions

An analytical solution has been obtained for the displacement and rotation angle of a single-joint flexible robot modeled by a continuous, clamped-free beam attached to a rigid hub of negligible radius. The results for a representative-applied harmonic torque are obtained and show the effect of the beam flexibility, which can be superimposed on the rigid-beam motion. Fourier series representations can be applied in the usual way to generalize these results. The nondimensionalization for both dynamic equations permitted a simpler problem formulation. The solution also provides a quick means of testing the effect of the various physical parameters on the design of a flexible one-link robot. The solution gives the accurate location of every point of the link and, thus, changing the parameters can give the desired results for a specific task for the arm.

Acknowledgments

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Enhancement of Data Separability in Multisensor-Multitarget Tracking Problems

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Introduction

In many current tracking problems, the tracking scenario may involve multiple targets whose states are to be estimated with observations from several sensors. The various aspects of the multitarget problems and the outlines of different solution techniques are given in Ref. 1.

In this Note, improvements to a clustering analysis-based modular approach[2] are made through the use of observation residuals. In such an approach, the multitarget problem is reduced to several single-target problems.

Simulation Models

The mathematical models of motion of the targets and the observations are presented here. The target motion is confined to two dimensions only. The scenarios considered here consist of moving targets and stationary observers.

System Model

The state vector consists of the reference Doppler frequency, f_0, the position components, x and y, velocity components, $\dot{x}$ and $\dot{y}$, and the acceleration components, a_x and a_y, of the target in an inertial x-y coordinate frame. The state vector is given by

$$X = [f_0, x, y, \dot{x}, \dot{y}, a_x, a_y]^T \tag{1}$$

The Doppler frequency and the target acceleration are modeled as constants driven by white noise processes. Consequently, the differential equation of motion of the target is

$$\dot{X} = AX + W \tag{2}$$

where

$$A = \left[\begin{array}{c|c} & 0_{1\times 4} \\ 0_{7\times 3} & I_{4\times 4} \\ & 0_{2\times 4} \end{array}\right] \tag{3}$$

and

$$W = [w_f, 0, 0, 0, 0, w_x, w_y]^T \tag{4}$$

w_f, w_x, and w_y are assumed to be zero-mean white Gaussian processes with power spectral densities of q_f, q_x, and q_y, respectively.

Measurement Model

In this study, the three components of the measurement vector, y, are the Doppler frequency, f, the sine of the bearing angle, and the cosine of the bearing angle measured between the signal source/sensor line and the reference x direction.[3]

The relation between the frequency shift and the relative speed of a signal source moving with respect to the sensor is

$$y_1 = f = \frac{f_0}{1 + v_R/c} \tag{5}$$

where f is the received frequency, f_0 the transmitted frequency, v_R the relative velocity of the source with respect to the sensor, and c the speed of sound in the propagated medium. The relative velocity, v_R, is given by $v_R = [(x - x_s)\dot{x} + (y - y_s)\dot{y}]/R$, where x_s and y_s are the sensor coordinates in the inertial x-y coordinates and $R = [(x - x_s)^2 + (y - y_s)^2]^{\frac{1}{2}}$. The bearing angle θ is defined as the angle between the x axis and the line joining the sensor and the target. The other two measurements are

$$y_2 = \sin\theta = (y - y_s)/R \tag{6}$$

and

$$y_3 = \cos\theta = (x - x_s)/R \tag{7}$$

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